DIFFERENTIAL PARAMETERS OF THE FIRST ORDER*

BY

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The general expression of a differential parameter of the first order is given in the symbolic representation † by

$$(U' \cdots U^{\lambda} f^{\lambda} \cdots f^{n-\lambda})(V' \cdots V^{\lambda} f' \cdots f^{n-\lambda}),$$

where $U', \cdots, U^{\lambda}$ and $V', \cdots, V^{\lambda}$ are functions of $x_1, \cdots, x_n$, and $f', \cdots, f^{n-\lambda}$ are equivalent symbols of the differential quadratic quantic

$$ds^2 = \sum_{i,\,k=1}^{n} a_{ik}\, dx_i\, dx_k .$$

These differential parameters and in particular the equations obtained by putting them equal to zero play an important rôle in all questions connected with the study of the orthogonality of directions in higher spaces.

In § 1 I set up four fundamental theorems concerning determinants, using ordinary (not symbolic) notation. In § 2 the symbolic method is applied to the construction of four important formulas which are used in the sequel and furnish at the same time numerous relations between differential parameters. In § 3 the directions orthogonal to all directions in a space of λ dimensions given by the equations $U', \cdots, U^{n-\lambda} = \text{const.}$ are determined and § 4 contains a general investigation of the conditions under which one space $V', \cdots, V^{n-\mu} = \text{const.}$ contains directions which are orthogonal to all directions of another space $U', \cdots, U^{n-\lambda} = \text{const.}$, and the determination of these directions.

§ 1. *Theorems on determinants.*

Designate the determinant $|x_i^k|$ of any μ^2 quantities x, where x_i^k is the element occupying the place in the ith row and kth column, by

$$(x' x^2 \cdots x^{\mu});$$

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† The symbolic method has been explained by me in the paper *A symbolic treatment of the theory of invariants of quadratic differential quantics of n variables;* Transactions of the American Mathematical Society, vol. 4 (1903), pp. 441–469. I shall quote this paper by I.

let us further use the abbreviation

$$(x' x^2 \cdots x^\lambda u' u^2 \cdots) = (x' x^2 \cdots x^\lambda u),$$

where the number of elements $u', u^2, \cdots$ is any.

1) If we form the determinant of degree $n+1$ the first n rows of which are

$$a_1' a_1^2 \cdots a_1^\lambda u_1' u_1^2 \cdots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n' a_n^2 \cdots a_n^\lambda u_n' u_n^2 \cdots,$$

where the elements a_k^i and u_k^i are any arbitrary quantities, then the determinant will vanish if we take for the $n+1$th row the elements

$$(a' b^2 \cdots b^{\lambda-1} u), \ (a^2 b^2 \cdots b^{\lambda-1} u), \ \cdots, \ (a^\lambda b^2 \cdots b^{\lambda-1} u),$$
$$(u' b^2 \cdots b^{\lambda-1} u), \ (u^2 b^2 \cdots b^{\lambda-1} u), \ \cdots,$$

where again the quantities $b_k^2, \cdots, b_k^{\lambda-1}$ are any. But all the terms $(u^k b^2 \cdots b^{\lambda-1} u)$ vanish. Thus, expanding the determinant according to its elements of the $n+1$th row, we have

$$\sum_{k=1}^{\lambda} (-1)^{k+1} (a^k b^2 \cdots b^{\lambda-1} u)(a' a^2 \cdots a^{k-1} a^{k+1} \cdots a^\lambda u) = 0,$$

or, by putting

$$a^\lambda = b',$$

$$\sum_{k=1}^{\lambda-1} (-1)^{k+1} (a^k b^2 \cdots b^{\lambda-1} u)(a' a^2 \cdots a^{k-1} a^{k+1} \cdots a^{\lambda-1} b' u) = (-1)^\lambda (a' a^2 \cdots a^{\lambda-1} u)(b' b^2 \cdots b^{\lambda-1} u).$$

If finally we write λ instead of $\lambda - 1$ and place b' between a^{k-1} and a^{k+1}, we obtain the formula

$$(1) \quad \sum_{k=1}^{\lambda} (a^k b^2 \cdots b^\lambda u)(a' a^2 \cdots a^{k-1} b' a^{k+1} \cdots a^\lambda u) = (a' a^2 \cdots a^\lambda u)(b' b^2 \cdots b^\lambda u).$$

This formula is identically true for all the involved quantities a_k^i, b_k^i and u_k^i.

2) We extend formula (1) by putting $(\mu < \lambda)$

$$a^{\mu+k} = c^k \qquad (k = 1, \cdots, \lambda - \mu),$$

and performing the summation first from 1 to μ, then from $\mu + 1$ to λ. Thus we have

$$(2) \quad \begin{aligned} &\sum_{k=1}^{\mu} (a^k b^2 \cdots b^\lambda u)(a' \cdots a^{k-1} b' a^{k+1} \cdots a^\mu c' \cdots c^{\lambda-\mu} u) \\ &\quad + \sum_{k=1}^{\lambda-\mu} (c^k b^2 \cdots b^\lambda u)(a' \cdots a^\mu c' \cdots c^{k-1} b' c^{k+1} \cdots c^{\lambda-\mu} u) \\ &\quad = (a' \cdots a^\mu c' \cdots c^{\lambda-\mu} u)(b' \cdots b^\lambda u) \end{aligned}$$

3) I wish to prove the following theorem

$$
\begin{vmatrix}
(p'a^2\cdots a^\lambda u), (a'p'\cdots a^\lambda u), \cdots, (a'a^2\cdots a^{i-1}p'a^{i+1}\cdots a^\lambda u), \cdots, (a'a^2\cdots a^{\lambda-1}p'u) \\
(p^2a^2\cdots a^\lambda u), (a'p^2\cdots a^\lambda u), \cdots, (a'a^2\cdots a^{i-1}p^2a^{i+1}\cdots a^\lambda u), \cdots, (a'a^2\cdots a^{\lambda-1}p^2u) \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
(p^\lambda a^2\cdots a^\lambda u), (a'p^\lambda\cdots a^\lambda u), \cdots, (a'a^2\cdots a^{i-1}p^\lambda a^{i+1}\cdots p^\lambda u), \cdots, (a'a^2\cdots a^{\lambda-1}p^\lambda u)
\end{vmatrix}
$$

$$
(3) \qquad = (p'p^2\cdots p^\lambda u)(a'a^2\cdots a^\lambda u)^{\lambda-1},
$$

where all the quantities a_k^i, p_k^i, u_k^i are arbitrary.

We prove by induction from $\lambda - 1$ to λ.

Every term of the minor P^k of the term $(p^k a^2 \cdots a^\lambda u)$ in the original determinant P contains the element a' which can be placed just in front of the elements u. The determinant P^k is then of the same type as P, with $\lambda - 1$ instead of λ, and a', u instead of u, and since the theorem is supposed to be true for determinants of order $\lambda - 1$, we have

$$P^k = (-1)^{k+1+(\lambda-1)^2}(p'\cdots p^{k-1}p^{k+1}\cdots p^\lambda a'u)(a^2\cdots a^\lambda a'u)^{\lambda-2},$$

or

$$P^k = (p'\cdots p^{k-1}a'p^{k+1}\cdots a^\lambda u)(a'a^2\cdots a^\lambda u)^{\lambda-2},$$

and

$$P = (a'a^2\cdots a^\lambda u)^{\lambda-2}\sum_{k=1}^{\lambda}(p^k a^2\cdots a^\lambda u)(p'\cdots p^{k-1}a'p^{k+1}\cdots p^\lambda u).$$

To this sum we can directly apply formula (1) and obtain

$$P = (p'\cdots p^\lambda u)(a'\cdots a^\lambda u)^{\lambda-1}.$$

Hence the theorem is generally true, since it is true for $\lambda = 2$, the equation

$$(p'a^2u)(a'p^2u) + (p^2a^2u)(p'a'u) = (p'p^2u)(a'a^2u)$$

being a special case of (1).

4) Let ρ, v', v^2, $\cdots$, v^λ, u', u^2, $\cdots$, $u^{n-\lambda}$ be any functions of the n independent variables x_1, x_2, $\cdots$, x_n, and define further

$$\rho_k = \frac{\partial\rho}{\partial x_k}, \qquad v_k^i = \frac{\partial v^i}{\partial x_k}, \qquad u_k^i = \frac{\partial u^i}{\partial x_k},$$

then I wish to prove the following theorem on Jacobians

$$(4)\quad (\rho v', \rho v^2, \cdots, \rho v^\lambda, u) = \rho^\lambda(v'v^2\cdots v^\lambda u) + \rho^{\lambda-1}\sum_{k=1}^{\lambda} v^k(v'\cdots v^{k-1}, \rho, v^{k+1}\cdots v^\lambda u),$$

where on the left side the general element of any one of the first λ columns is $\partial\rho v^i/\partial x_k$, the notation otherwise being the same as in the preceding work. We prove again by induction from λ to $\lambda + 1$.

We have directly

$$(\rho v', \cdots, \rho v^\lambda, \rho v^{\lambda+1}, u) = \rho(\rho v', \cdots, \rho v^\lambda, v^{\lambda+1}, u) + v^{\lambda+1}(\rho v', \cdots, \rho v^\lambda, \rho, u),$$

but from (4), by putting the first one of the quantities u equal to $v^{\lambda+1}$,

$$(\rho v', \cdots, \rho v^\lambda, v^{\lambda+1}, u) = \rho^\lambda(v' \cdots v^{\lambda+1} u) + \rho^{\lambda-1}\sum_{k=1}^{\lambda} v^k(v' \cdots v^{k-1}\rho v^{k+1} \cdots v^{\lambda+1} u)$$

and similarly

$$(\rho v', \cdots, \rho v^\lambda, \rho, u) = \rho^\lambda(v' \cdots v^\lambda \rho u) + \rho^{\lambda-1}\sum_{k=1}^{\lambda} v^k(v' \cdots v^{k-1}\rho v^{k+1} \cdots v^\lambda \rho u).$$

Each one of the terms of the last sum vanishes, hence

$$(\rho v', \cdots \rho v^{\lambda+1}, u) = \rho^{\lambda+1}(v' \cdots v^{\lambda+1} u) + \rho^\lambda \sum_{k=1}^{\lambda} v^k(v' \cdots v^{k-1}\rho v^{k+1} \cdots v^{\lambda+1} u) + \rho^\lambda(v' \cdots v^\lambda \rho u),$$

or

$$(\rho v', \cdots, \rho v^{\lambda+1}, u) = \rho^{\lambda+1}(v' \cdots v^{\lambda+1} u) + \rho^\lambda \sum_{k=1}^{\lambda+1} v^k(v' \cdots v^{k-1}\rho v^{k+1} \cdots v^{\lambda+1} u).$$

The theorem is true for $\lambda = 1$, namely

$$(\rho v, u) = \rho'(v, u) + \rho^0 v(\rho, u),$$

hence it is generally true.

§ 2. *Relations between differential parameters.*

5) In formula (2) we omit the quantities u, put $\lambda = n$ and write λ instead of μ. We let further $c', \cdots, c^{n-\lambda}$ be symbols of a quadratic differential quantic of n variables and designate them by $f', \cdots, f^{n-\lambda}$. We finally multiply every term of (2) by some alternating function $[f', \cdots, f^{n-\lambda}] = [f]$ of $f', \cdots, f^{n-\lambda}$. Then the general term of the second sum

$$(f^k b^2 \cdots b^n)(a' \cdots a^\lambda f' \cdots f^{k-} \; b' f^{\;+1} \cdots f^{n-\lambda})[f] = (f' b^2 \cdots b^n)(a' \cdots a^\lambda b' f^2 \cdots f^{n-\lambda})[f].$$

Hence all the terms of this sum are equal and we have

$$(5) \quad \sum_{k=1}^{\lambda}(a^k b^2 \cdots b^n)(a' \cdots a^{k-1} b' a^{k+1} \cdots a^\lambda f)[f] = (a' \cdots a^\lambda f)(b' \cdots b^n)[f] - (n-\lambda)(f' b^2 \cdots b^n)(a' \cdots a^\lambda b' f^2 \cdots f^{n-\lambda})[f].$$

6) In (5) we put

$$b' = Q;\; b^2, \cdots, b^n = \phi^2, \cdots, \phi^n \text{ (symbols)},$$
$$a', \cdots, a^\lambda = V', \cdots, V^\lambda,$$

take

$$[f] = (U' \cdots U^\lambda f),$$

and multiply by $(P\phi)$, then the second term on the right side of (5) can be transformed by means of I(34), namely

$$(f'\phi)(P\phi)(U'\cdots U^{\lambda}f'f^2\cdots f^{n-\lambda})=(n-1)!(U'\cdots U^{\lambda}Pf),$$

so that we obtain the following formula

$$(P\phi)(U'\cdots U^{\lambda}f)\sum_{k=1}^{\lambda}(V_k\phi)(V'\cdots V^{k-1}QV^{k+1}\cdots V^{\lambda}f)$$

$$(6)\qquad =(P\phi)(Q\phi)(U'\cdots U^{\lambda}f)(V'\cdots V^{\lambda}f)$$
$$-(n-\lambda)(n-1)!\,(U'\cdots U^{\lambda}Pf)(V'\cdots V^{\lambda}Qf).$$

7) If in (6) we put

$$Q=U'=\psi\,(\text{symbol}),$$

then we can apply to the first term on the right side the transformation I (34) so that

$$(P\phi)(\psi\phi)(\psi U^2\cdots U^{\lambda}f)=(n-1)!\,(PU^2\cdots U^{\lambda}f).$$

The two terms on the right side of (6) combine then to

$$(n-\lambda+1)(n-1)!\,(PU^2\cdots U^{\lambda}f)(V'\cdots V^{\lambda}f),$$

and by writing U' instead of P we obtain with a slight change the formula

$$(U\phi)(U^2\cdots U^{\lambda}f)\sum_{k=1}^{\lambda}(-1)^{k+1}(V_k\phi)(V'\cdots V^{k-1}V^{k+1}\cdots V^{\lambda}f)$$

$$(7)\qquad =(n-\lambda+1)(n-1)!\,(U'\cdots U^{\lambda}f)(V'\cdots V'f).$$

8) I wish to prove that

$$\begin{vmatrix}(U'f') & \cdots & (U^mf')\\ \cdot & \cdots & \cdot\\ (U'f^m) & \cdots & (U^mf^m)\end{vmatrix}\prod_{i=1}^{m}(V^if^i)=c_m(U'\cdots U^mf)(V'\cdots V^mf),$$

where every f^k on the left side stands for a set of $n-1$ equivalent symbols and where c_m is a numerical constant depending on m. Denote the left side of the above equation by R.

I prove by induction from $m-1$ to m. Then we have

$$\begin{vmatrix}(U'f^2)\cdots(U^{k-1}f^2)(U^{k+1}f^2)\cdots(U^mf^2)\\ \cdot\quad\cdots\quad\cdot\\ (U'f^m)\cdots(U^{k-1}f^m)(U^{k+1}f^m)\cdots(U^mf^m)\end{vmatrix}\prod_{i=2}^{m}(V^if^i)$$
$$=c_{m-1}(U'\cdots U^{k-1},\ U^{k+1}\cdots U^mf)(V^2\cdots V^mf),$$

and if we develop the determinant-factor of R according to its elements of the first row

$$R = c_{m-1}(V'f')(V^2 \cdots V^m f) \sum_{k=1}^{m} (-1)^{k+1}(U^k f')(U' \cdots U^{k-1} U^{k+1} \cdots U^m f),$$

which reduces by means of (7) to

$$R = (n - m + 1)(n - 1)!\, c_{m-1}(U' \cdots U^m f)(V' \cdots V^m f).$$

For $m = 1$ the theorem is true, and $c_1 = 1$. Hence the theorem is generally true. The constant c_m is easily determined, namely

$$c_m = (n-1)(n-2)\cdots(n-m+1)[(n-1)!]^{m-1},$$

so that

$$\text{(8)} \qquad \begin{vmatrix} (U'f') \cdots (U^m f') \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (U'f^m) \cdots (U^m f^m) \end{vmatrix} \prod_{i=1}^{m} (V^i f^i)$$

$$= \frac{[(n-1)!]^m}{(n-m)!}(U' \cdots U^m f)(V' \cdots V^m f).$$

§ 3. *Determination of directions orthogonal to all directions in a given space U.*

We define in a general space R_n of n dimensions whose arc-element is determined by

$$\text{(9)} \qquad ds^2 = \sum_{r,\, s=1}^{n} a_{rs}\, dx_r\, dx_s$$

a surface (space) of λ dimensions, R_λ, by the $n - \lambda$ equations

$$\text{(10)} \qquad U' = \text{const.}, \ \cdots, \ U^{n-\lambda} = \text{const.},$$

where the $n - \lambda$ quantities U are functions of $x_1, \cdots, x_n$.

The n differentials $dx_1, \cdots, dx_n$ satisfying the $n - \lambda$ equations

$$\text{(11)} \qquad \sum_{r=1}^{n} U'_r\, dx_r = 0, \ \cdots, \ \sum_{r=1}^{n} U^{n-\lambda}_r\, dx_r = 0,$$

define a certain direction in R_λ. We agree, as always in the following, to denote differentiation with respect to x_r by the lower index r.

In order to solve the equations (11) symmetrically we introduce λ arbitrary functions $V', \cdots, V^\lambda$ with the restriction, however, that the Jacobian

$$\text{(12)} \qquad D = (V' \cdots V^\lambda U' \cdots U^{n-\lambda}) \neq 0.$$

If the minors of the element V_r^k in D are denoted by A^{kr}, it is clear that the λ systems of differentials

$$dx_1^k = \rho A^{k1}, \cdots, dx_n^k = \rho A^{kn} \qquad (k = 1, \cdots, \lambda) \tag{13}$$

satisfy equations (11) and represent λ independent (i. e., not contained in a space of less than λ dimensions) directions in R_λ. The general direction in R_λ — general solution of (11) — is then given by

$$dx_1 = \sum_{k=1}^{\lambda} \rho_k A^{k1}, \cdots, dx_n = \sum_{k=1}^{n} \rho_k A^{kn}, \tag{14}$$

where $\rho_1, \cdots, \rho_\lambda$ are arbitrary parameters.

If p is any arbitrary quantity we have

$$\sum_{r=1}^{n} p_r dx_r = \sum_{k=1}^{\lambda} \rho_k (V' \cdots V^{k-1} p V^{k+1} \cdots V^\lambda U), \tag{15}$$

and this expression serves conveniently to define the direction $dx_1, \cdots, dx_n$ as the ratios of the coefficients of $p_1, \cdots, p_r$. We shall simply call it the direction p.

The condition that two directions defined by two systems dx and δx are to be perpendicular to each other in R_n is

$$\sum_{r, s=1}^{n} a_{rs} dx_r \delta x_s = 0,$$

or, in symbolic notation,

$$\sum_{r=1}^{n} f_r dx_r \cdot \sum_{s=1}^{n} f_s dx_s = 0. \tag{16}$$

If now the direction δx in R_n is to be perpendicular to all the directions dx in R_n it will be necessary and sufficient that δx is perpendicular to the λ independent directions defined by (13).

Hence

$$(V' \cdots V^{k-1} f V^{k+1} \cdots V^\lambda U) \sum_{s=1}^{n} f_s \delta x^s = 0 \tag{17}$$

for every $k = 1, \cdots, \lambda$.

To solve these equations we adjoin again $n - \lambda$ arbitrary functions $W', \cdots, W^{n-\lambda}$ and form the determinant Δ of the derivatives of the quantities W with respect to $x_1, \cdots, x_n$ and the coefficients of the differentials δx_s in (17). We obtain

$$\Delta = (W' \cdots W^{n-\lambda} f' \cdots f^\lambda) \prod_{k=1}^{\lambda} (V' \cdots V^{k-1} f V^{k+1} \cdots V^\lambda U)$$

and if we permute the equivalent symbols $f', \cdots, f^\lambda$ in all possible ways, add together and divide by $\lambda!$,

$$\Delta = \frac{1}{\lambda!}(f' \cdots f^\lambda W)$$

$$\times \begin{vmatrix} (f' V^2 \cdots V^\lambda U) \cdots (V' \cdots V^{k-1} f' V^{k+1} \cdots V^\lambda U) \cdots (V' \cdots V^{\lambda-1} f' U) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (f^\lambda V^2 \cdots V^\lambda U) \cdots (V' \cdots V^{k-1} f^\lambda V^{k+1} \cdots V^\lambda U) \cdots (V' \cdots V^{\lambda-1} f' U) \end{vmatrix},$$

which reduces according to (3) to

$$\Delta = \frac{1}{\lambda!}(f' \cdots f^\lambda W)(f' \cdots f^\lambda U)(V^\lambda \cdots V^\lambda U)^{\lambda-1}.$$

Denoting now the minor of the element W_r^a in Δ by B^{ar} we find the following $n - \lambda$ systems of solutions of (17) by taking the factor

$$\frac{1}{\lambda!}(V' \cdots V^\lambda U)^{\lambda-1}$$

into the proportionality-factor σ,

$$\delta x_1^a = \sigma B^{a1}(f' \cdots f^\lambda U), \cdots, \delta x_n^a = \sigma B^{an}(f' \cdots f^\lambda U) \qquad (a=1, \cdots, n-\lambda).$$

The general expression of any direction δx in R_n perpendicular to every direction in R_λ is therefore given by

$$\delta x_r = (f' \cdots f^\lambda U) \sum_{a=1}^{n-\lambda} \sigma_a B^{ar} \qquad (r=1, \cdots, n),$$

or, using the notation explained in (15),

$$(19) \quad \sum_{r=1}^{n} q_r \delta x_r = (U' \cdots U^{n-\lambda} f) \sum_{a=1}^{n-\lambda} \sigma_a (W' \cdots W^{a-1} q W^{a+1} \cdots W^{n-\lambda} f),$$

where $\sigma_1 \cdots \sigma_{n-\lambda}$ are $n - \lambda$ arbitrary parameters.

By a proper selection of these parameters this expression can be considerably simplified. I put

$$\sigma_a = (W^a \phi)(U^k \phi),$$

where the $n - 1$ quantities ϕ are again symbols of (9).

To (19) we can directly apply formula (6) where now the term

$$(U' \cdots U^{n-\lambda} P f)$$

vanishes for $P = U^k$ so that we have

$$\sum_{r=1}^{n} q_r dx_r = (q\phi)(U^k \phi)(U' \cdots U^{n-\lambda} f)(W' \cdots W^{n-\lambda} f),$$

and since we are only concerned with the ratios of the differentials dx we can omit the factor

$$(U' \cdots U^{n-\lambda} f)(W' \cdots W^{n-\lambda} f)$$

which admits of actual (not only symbolical) interpretation and have now the result

$$(20) \qquad \sum_{r=1}^{n} q_r \delta x_r^k = \rho (U^k f)(qf) \qquad (k=1, \cdots, n-\lambda),$$

which defines $n - \lambda$ independent directions in R_n, all of which are perpendicular to the general direction p (15) in R_λ. From (20) we form the general direction in R_n perpendicular to R_λ, namely,

$$(21) \qquad \sum_{r=1}^{n} q_r \delta x_r = \sum_{k=1}^{n-\lambda} \rho_k (U^k f)(qf),$$

where $\rho_1, \cdots, \rho_{n-\lambda}$ are arbitrary parameters. Thus we have the following

THEOREM I. *Given a space of λ dimensions R_λ by the $n - \lambda$ equations*

$$U' = const., \cdots, U^{n-\lambda} = const.,$$

contained in the general space R_n of n dimensions defined by (9). *Then every direction $\delta x_1 : \delta x_2 : \cdots : \delta x_n$ in R_n which is normal at a point P of R_λ to every possible direction in R_λ at P is given by the ratios of the coefficients $q_1 : q_2 : \cdots : q_n$ in the expression* (21).

A posteriori it is easily verified that the values of δx taken from (21) or (20) satisfy equation (17), for

$$\sum_{r=1}^{n} f_r \delta x_r^i = \rho (U^i \phi)(f\phi),$$

and $(U^i \phi)(f\phi)(V' \cdots V^{k-1} f V^{k+1} \cdots V^\lambda U) = 0$ from I (34) for every value of i and k.

§ 4. *General study of the case where one space V contains directions orthogonal to all directions of another space U.*

If a space V of μ dimensions

$$(22) \qquad V', V^2, \cdots, V^{n-\mu} = \text{const.}$$

contains a direction which is normal to all the directions of a space U of λ dimensions

$$(23) \qquad U', U^2, \cdots, U^{n-\lambda} = \text{const.},$$

then the $n - \mu$ equations

$$\sum_{r=1}^{n} V_r^i dx_r = 0 \qquad (i=1, \cdots, n-\mu)$$

must be satisfied by substituting for $dx_1, \cdots dx_n$ the coefficients of q of the expression

$$(24) \qquad (q\phi) \sum_{k=1}^{n-\lambda} \phi_k (U^k \phi),$$

i. e., we must have

$$(25)\qquad (V^i\phi)\sum_{k=1}^{n-\lambda}\rho_k(U^k\phi)=0 \qquad (i=1,\cdots,n-\mu).$$

Every system of values ρ_k satisfying these $n-\mu$ homogeneous linear equations gives us one direction q of the required property.

We treat first the case where the number of equations is less than the number of unknowns ρ_k, i. e.,

$$n-\mu<n-\lambda \qquad \text{or} \qquad \mu>\lambda.$$

Then the general solution of (25) is

$$(26)\quad \rho_k=(-1)^{k+1}(U'\cdots U^{k-1}U^{k+1}\cdots U^{n-\lambda}f)(V'\cdots V^{n-\mu}\omega'\cdots\omega^{\mu-\lambda-1}f),$$

where $\omega',\cdots,\omega^{\mu-\lambda-1}$ are arbitrary quantities. Indeed the left side of (25) reduces by means of (7) to

$$(U'\cdots U^{n-\lambda}f)(V^iV'\cdots V^{n-\mu}\omega'\cdots\omega^{\mu-\lambda-1}f),$$

which vanishes for $i=1,\cdots,n-\mu$.

We obtain the direction q itself by substituting the value of ρ_k (26) into (24) which gives, again according to (7),

$$(27)\qquad (qV'\cdots V^{n-\mu}\omega\cdots\omega^{\mu-\lambda-1}f)(U'\cdots U^{n-\lambda}f).$$

Hence we have

THEOREM II. *If a space U defined by $U',\cdots,U^{n-\lambda}=$ const., and a space V defined by $V',\cdots,V^{n-\mu}=$ const., have at least one point P in common, then there exists, provided that $\mu>\lambda$, at every point P always and in general (i. e., if no special relations between U and V hold) only $\infty^{\mu-\lambda-1}$ (i. e., $\mu-\lambda$ independent) directions in V which are perpendicular to all directions in U through P. These directions are given by the coefficients of $q_1\cdots q_n$ in the expression* (27).

If there shall exist more than $\mu-\lambda$ independent directions in V normal to U, then the rank* of the matrix of the coefficients of ρ_k in (25) must be $<n-\mu$. If the rank is $n-\mu-s$, then there are $\mu-\lambda+s$ independent directions. Putting in this case for abbreviation

$$n-\mu-s+1=\alpha,$$

we see that every determinant

$$(V^{i_1}f')\cdots(V^{i_\alpha}f^\alpha)\begin{vmatrix}(U^{k_1}f')\cdots(U^{k_\alpha}f')\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ (U^{k_1}f^\alpha)\cdots(U^{k_\alpha}f^\alpha)\end{vmatrix}$$

* A matrix is of rank r if all its determinants of degree $r+1$ vanish, but not all its determinants of degree r.

must vanish for all values $i_1, \cdots, i_\alpha = 1, \cdots, n-\mu$, and $k_1, \cdots, k_\alpha = 1, \cdots n-\lambda$. Applying now to this determinant formula (8) we have the conditions

$$(U^{k_1} \cdots U^{k_\alpha} f)(V^{k_1} \cdots V^{i_\alpha} f) = 0.$$

If these conditions are satisfied the quantities ρ_k are determined by any (properly chosen) $n - \mu - s$ of the equations (25). We find

$$\rho_k = (-1)^{k+1}(U' \cdots U^{k-1} U^{k+1} \cdots U^{n-\lambda} f)(V^{i_1} \cdots V^{i_{\alpha-1}} \omega' \cdots \omega^{n-\lambda-\alpha} f)$$

and the required $\mu - \lambda + s$ independent directions as the coefficients of $q_1, \cdots, q_n$ in

$$(q V^{i_1} \cdots V^{i_{\alpha-1}} \omega' \cdots \omega^{\mu-\lambda+s-1} f)(U' \cdots U^{n-\lambda} f).$$

We have, then, the following

THEOREM III. *If a space V (22) and U (23), where $\mu > \lambda$, have at least one point P in common, then the necessary and sufficient conditions that V at every point P contains $\mu - \lambda + s$ independent directions which are perpendicular to all directions U through P are*

$$(U^{k_1} \cdots U^{k_\alpha} f)(V^{i_1} \cdots V^{i_\alpha} f) = 0,$$

for every set of values

$$k_1, \cdots, k_\alpha = 1, \cdots, n - \lambda,$$

and

$$i_1, \cdots, i_\alpha = 1, \cdots, n - \mu,$$

where

$$\alpha = n - \mu - s + 1.$$

These directions are given by the coefficients of $q_1, \cdots, q_n$ in

$$(q V^{i_1} \cdots V^{i_{n-\mu-s}} \omega' \cdots \omega^{\mu-\lambda+s-1} f)(U' \cdots U^{n-\lambda} f).$$

Applying now similar methods to the discussion of the remaining cases $\mu = \lambda$ and $\mu < \lambda$, always using equations (7) and (8) for reduction, we are led to the following theorems.

THEOREM IV. *If two spaces of equal dimensions $U', \cdots, U^{n-\lambda} = const.$, and $V', \cdots, V^{n-\lambda} = const.$ have at least one point in common, then the necessary and sufficient condition that V contains at every point P one direction normal to all directions in U through P is*

$$(U' \cdots U^{n-\lambda} f)(V' \cdots V^{n-\lambda} f) = 0.$$

This direction is given by the coefficients of $q_1 \cdots q_n$ in

$$(q V' \cdots V^{k-1} V^{k+1} \cdots V^{n-\lambda} f)(U' \cdots U^{n-\lambda} f),$$

where k is arbitrary. In this case there is also one direction in U which is normal to every direction in V; it is given by

$$(qU' \cdots U^{k-1}U^{k+1} \cdots U^{n-\lambda}f)(V' \cdots V^{n-\lambda}f).$$

THEOREM V. *If a space V (22) and U (23), where $\mu \leqq \lambda$, have at least one point P in common, then the necessary and sufficient conditions that V at every point P contains s independent directions which are normal to all directions in U through P are*

$$(V^{i_1} \cdots V^{i_\beta}f)(U^{k_1} \cdots U^{k_\beta}f) = 0$$

for every set of values

$$i_1, \cdots, i_\beta = 1, \cdots, n-\mu,$$

$$k_1, \cdots, k_\beta = 1, \cdots, n-\lambda,$$

where

$$\beta = n - \lambda - s + 1.$$

These directions are given by the coefficients of $q_1 \cdots q_n$ in the expression

$$(qV^{i_1} \cdots V^{i_{n-\lambda-s}}\omega' \cdots \omega^{s-1}f)(U' \cdots U^{n-\lambda}f).$$

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